Mathematics is just more language

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Overview

- Mathematics is hybrid: it is both an object of thought and a tool of thought.
- Late 20th century analytic philosophy focused almost exclusively on mathematics as “object of thought”, i.e. something we have theories *about*.
- If mathematical stuff is just additional stuff to think *about*, then Ockhamist considerations would suggest trying to do without it.
- If we are thinking *with* mathematics, then it is no more dispensable than concepts or language.

A narrative

History of the philosophy of applied mathematics

- Pre 19th century, through Kant
  I imagine that it didn't occur to anyone that commitment to mathematical theories was optional.
- Non-euclidean geometries
- Conventionalism

History of the philosophy of applied mathematics

- Carnap's frameworks
- Quine's package deal
- Field's program

History of the philosophy of applied mathematics

- HH: The absurdity to which Field's program leads uncovers something deeply wrong at the heart of Quine's view.
- HH: Quine errs in attempting to describe everything from a third-person point of view.
- But I will leave that here — it's murky philosophy that we can avoid today.

“Mathematics is not to be regarded as a special branch of knowledge based on the accumulation of experience but rather as a **refinement of general language**, supplementing it with appropriate tools to represent relations for which ordinary verbal expression is imprecise or too cumbersome.” (Niels Bohr)

Field's program

The Putnam-Quine indispensability argument

- The ideally rational person accepts the best scientific theories and all the things that they are committed to.
- (Indispensability) Our best scientific theories cannot dispense with mathematical objects.
- The ideally rational person is a mathematical Platonist.

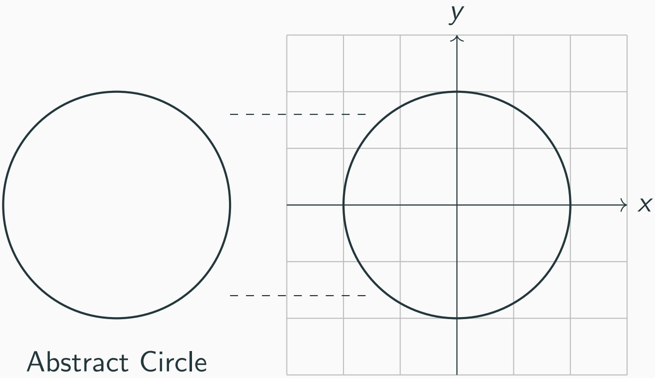
Two responses

- Fictionalism: Reject the first premise, you don't have to take all those things literally.
- Eliminativism: Reject the second premise by showing that our best scientific theories can dispense with mathematical objects.
  Hartry Field: Replace the Platonistic T with a nominalistic version $N(T)$.

“If they do not exist, should not it be possible, at least in principle, to characterize the physical world without talking about them at all? Here is another job for philosophers: to find alternatives to standard ‘platonistic’ (mathematical-entity-invoking) physical theories which can do the same empirical explanatory work without requiring any mathematical entities to exist.” (Arntzenius and Dorr, 2012)

Motivating example

Field takes the relationship between numerical and synthetic geometry as a paradigm example of nominalization.



Circle in $\mathbb{R}^2$

Worries about Field's program

Nominalizing by fiat

- For any theory T, interpret all terms as applying to some kind of "physical" stuff.
- Arntzenius: We use fiber bundles in physics, so let us assume that elements of a fiber bundle are real physical things.
- Schroeren: We use symmetry groups in physics, so let us assume that the group elements are real physical things.

“The mathematics of phase space is part of the genuine content of classical mechanics. This bit of the mathematics used in our physics cannot be representationally inert: it is inherent to the dynamics. One way to spell this out further would be to be a substantivalist about statespace. Again, I happen to be open to the idea. But we need not go that way. The important thing is that phase space is not just a mathematical convenience. Some part of reality is more or less directly represented by it.” (North 2009)

Nominalizing by fiat

- Field takes **RCF** to be quantifying over numbers, but $\mathbf{E}^2$ to be quantifying over physical objects.
- I'm not interested in non-mathematical properties of theories.

Ontological commitment is overrated

- Autobiography: I wasn't committing to Platonism when I used a vector space to describe the sum of two forces.
- A Morita extension of T adds new domains of quantification.
- Morita equivalent theories need not agree on how many things there are.
- What's an object and what's vocabulary can depend on formulation.

Concrete mathematics is closer to reality

- Claim: Field has it backwards in thinking that synthetic geometry is closer to *Anschauung* than numerical geometry.
- Syntactically: Assigning physical extensions to terms in the signature of $Th(\mathbb{R}^2)$ is *easier* than assigning physical extensions to terms in the signature of $\mathbb{E}^2$.
- Syntactically: Assigning elements of $\mathbb{R}^2$ to points of space is *easier* than assigning elements of a model M of $\mathbb{E}^2$.

Concrete mathematics is closer to reality

- A perfect circle is a classic example of a Platonic object
- An extensionless point is a classic example of a Platonic object

The argument form

GTR is the best theory of gravitational phenomena.

GTR uses smooth manifolds.

Smooth manifolds live in a model of ZF set theory.

GTR entails the truth of ZF set theory.

The best theory of gravitational phenomena is committed to the existence of sets.

The argument form

GTR uses symbols such as "g_{ab}".

The best theory of gravitational phenomena is committed to the existence of symbols.

On quantifying over numbers or sets

In realistic examples, it is unclear what a mathematical theory quantifies over.

- A smooth manifold M has a collection of coordinate charts, each of which is a mapping to $\mathbb{R}^n$.
- A Hilbert space is a vector space over $\mathbb{C}$.
- A C^*-algebra is a set A with a function $\mathbb{C} \times A \to A$.
- The dual of a group G is the set of mappings of G into the circle group $\mathbb{T}$.

A meta-pragmatist manifesto

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- I want to know what difference our philosophical ruminations and theories might make for scientific practice.
- Example: The claim that "wave mechanics and matrix mechanics are equivalent" should end the search for experiments or arguments that would favor one of the two.
- Example: If Lagrangian and Hamiltonian mechanics are inequivalent, then physicists should change their practice.

- Claim: Something between Morita equivalence and categorical equivalence is the best proposal we have for scientific practice.
- This claim is not an attempt to describe (as “naturalists” would want), nor is it an attempt to regulate science with ariori philosophical insight. Rather, it is **explicative**.

Back to Wissenschaftslogik!

Carnap's call for objectivity

10 *a*. The sentences of arithmetic *state* (or: express) certain properties of numbers and certain relations between numbers.	10 *b*. The sentences of arithmetic are composed of numerical expressions and one- or many-termed numerical predicates combined in such and such a way.
11 *a*. A particular sentence of physics *states* the condition of a spatial point at a given time.	11 *b*. A particular sentence of physics consists of a descriptive predicate and spatio-temporal co-ordinates as arguments.

The following examples 12 *a*, 13 *a*, and 14 *a* appear at first to be of the same kind as 1 *a* and 4 *a*. Actually, however, they demonstrate particularly clearly the danger of error which is involved in the use of the material mode of speech.

12 *a*. This letter *is about* the son of Mr. Miller.	12 *b*. In this letter a sentence $\mathfrak{Pr}(\mathfrak{A}_1)$ occurs in which $\mathfrak{A}_1$ is the description 'the son of Mr. Miller'.
13 *a*. The expression 'le cheval de M' *designates* (or: means) the horse of M.	13 *b*. There is an equipollent expressional translation from the French into the English language in which 'the horse of M' is the correlate of 'le cheval de M'.
14 *a*. The expression 'un éléphant bleu' *means* a blue elephant.	14 *b*. (Analogous to 13 *b*.)

Non-mathematical properties of theories

- Neil Dewar is thinking about T
- T is about birds
- T is nominalistic
- T quantifies over mathematical objects

Mathematical properties of theories

- T is finitely axiomatizable
- T is $\aleph_0$-categorical
- There is a conservative translation $F : T \to T'$

Mathematical properties of theories

- We are most interested in properties that are invariant under the relevant notion of theoretical equivalence.
- For example, "T has the symbol 'P' in its signature" is not invariant.

Three perspectives on theories

- Syntax
  - Objects: A set of axioms in a formal language
  - Arrows: n-dimensional interpretations
- Semantics
  - Objects: A category of models with some (elusive!) additional structure
  - Arrows: Functors that preserve the additional structure

Three perspectives on theories

- Hybrid
    - Objects: Syntactic categories, e.g. coherent categories
    - Arrows: coherent functors

The 2-category of first-order theories

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$$\begin{tikzcd} T \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow"] & & T' \end{tikzcd}$$

The 2-category of first-order theories

- Definition: $F : T \to T'$ is conservative if $T' \vdash F\phi$ implies $T \vdash \phi$.
- Definition: $F : T \to T'$ is essentially surjective if for each $\psi \in \Sigma'$, there is a $\phi \in \Sigma$ such that $T' \vdash \psi \leftrightarrow F\phi$.
- How to define essentially surjective for *n*-dimensional translations?

Revamping Field's program

- Field: The desired result is that T is a **conservative extension** of $N(T)$.
- Analogy? $T = N(T) + \{c\}$
- HH: No, there is a more interesting relationship where T is like a covering space of $N(T)$
- Analogy? The relationship of an affine space to its underlying vector space

- Intuitively, there is a model of $\mathbf{E}^2$ in $\mathbb{R}^2$.
- There is a translation of $\mathbf{E}^2$ into **RCF**.
- Zayton: There is a more interesting relationship between **RCF** and $\mathbf{E}^2$ having to do with interpretations with parameters.

- Field was probably thinking of the Platonist theory as a theory T' with two sorts, physical objects and numbers, and with a function f from objects to numbers.
- But T' is Morita equivalent to $Th(\mathbb{R}^2)$.

Conclusion

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- What attitude should we take toward mathematical language that is employed in empirical science?
- I find the question to be too abstract. Compare with: what kind of attitude should I take to works of art?
- Hartry Field gives a concrete theory-building proposal: *replace Platonistic theories by nominalistic theories*.

Conclusion

- Field's proposal fails Carnap's concreteness test.
- What proposals do you have for improving our use of mathematical language in science?