Speaking about Models

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1. Most scientists and philosophers take set-theoretic foundations of math for granted — at roughly the level of Halmos' *Naive Set Theory*. They assume that the "niceties" of set theory won't make any difference to their research. And they assume that set theory is as safe as anything else we could put in the foundations.

2. Sometimes when we[1] get engaged in detailed investigations, set theory becomes relevant — and we have to decide how far we want to commit to it, and how we understand its consequences.

   [1] especially mathematicians, mathematical physicists, and technically oriented philosophers

   - When applicable results depend on choice axioms (e.g. Tarski-Banach, existence of singular distributions)
   - Putnam's paradox
   - The hole argument

3. This talk descends from two papers:

   - Dimitris Tsementzis and HH, "Foundations and Philosophy"
   - HH and JB Manchak, "Closing the Hole Argument"

   The first paper considers how replacing set theory (ZF) with homotopy type theory (HoTT) might change our philosophical logic. What's relevant for us is that HoTT tries to fix some problems that might result from a too literal reading of ZF. The second paper argues that the hole argument of GTR is based on a misunderstanding about how to reason about models (structures in the set-theoretic universe) that, once again, results from a too literal reading of ZF.

4. The basic puzzle: for models M and N of a theory, how are we to understand the situation where M and N are isomorphic but not identical?

5. The puzzle restated

   Suppose that $M \neq N$ but that $h : M \to N$ is an isomorphism.

   - Defn: M and N are *elementarily equivalent* just in case $M \vDash \phi$ iff $N \vDash \phi$.
   - Fact: If M and N are isomorphic, then M and N are elementarily equivalent.
   - There is a set-theoretic predicate Φ such that $\Phi(M)$ but $\neg\Phi(N)$.

6. Example
   - Let T be the empty theory in the empty language.
   - Let M be the model of T with domain $\{\emptyset\}$.
   - Let N be the model of T with domain $\{\{\emptyset\}\}$.
   - For a Σ-structure X, let Φ be the set-theoretic predicate $\Phi(X) \equiv (\emptyset \in |X|)$.
7. Is the availability of non-isomorphism invariant predicates a bug of set-theoretic foundations, or a feature?
   - This fact seems to drive the hole argument in GTR, as well as Putnam's paradox.
8. If it's a bug, can it (and should it) be fixed by changing foundations?
   - Hermeneutic structuralism versus revolutionary structuralism
9. Putnam's trick: Let M be a model of T, and let $h : N \to M$ be a bijection of sets. Define $r^N = h^{-1}(r^M)$ for each relation symbol r. Then the models M and N are isomorphic. But if h is non-trivial, then M and N disagree on the properties of individuals.
   - Hilbert, Burgess, Linnebo (hermeneutic structuralism): Pure mathematicians are trained to ignore the differences between M and N. It's not the fault of ZF if an untrained person takes M and N to represent different situations.
10. Does Putnam's trick also lie behind the "hole argument" of Einstein's general theory of relativity?
    - The hole argument is supposed to derive a bad consequence (pernicious indeterminism) from a reasonable ontological assumption (spacetime is a substance).
    - Einstein "Lochbetrachtung" 1913; Earman and Norton "What price spacetime substantivalism?" 1987
    - Simple analogy:
      - A model M of T consists of two sets M_0, M_1, each of which contains two elements, and a functional relation $r \subseteq M_0 \times M_1$ whose intended meaning is temporal development. i.e. we stipulate that for each $a \in M_0$ there is a unique $a' \in M_1$ such that $\langle a, a' \rangle \in r$.
      - Given a bijection $h : M_1 \to M_1$, the Putnam trick can be used to crete another model that is isomorphic to M.

  - The theory T thus fails the Montague-Lewis-Earman definition of determinism.

11. Benacerraf's antinomy: If $0, 1, 2, 3, \ldots$ are sets, then there are many facts about them that we have no means of discovering

  - Is $\emptyset \in 1$?

12. A middle way?

  - There are too many set-theoretic predicates: they detect differences that are *not* really there
  - There are too few Σ-sentences: they cannot detect real differences
    - Example: $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$ are elementarily equivalent.
    - Recall: In the case of finite structures, elementary equivalence implies isomorphism.

13. Open question: Can we explicate mathematicians' talk about "properties invariant under isomorphism" in terms of definable properties of models? This would suggest a *preferred language for speaking about models of a theory*.

  - Topological properties: Hausdorff, compact, metrizable, simply connected, etc.
  - Group properties: abelian, number of subgroups, number of normal subgroups, etc.
  - Relativistic spacetimes: singular, globally hyperbolic, etc.
    - The issue is of real importance here. E.g. it is easy to mistake coordinate singularities for genuine singularities. In fact, it's not so easy to check that proposed definitions of "is singular" are invariant under isomorphism

14. Some first ideas

  - Properties of models that are representable by first-order sentences
    - Partial orders: has a left endpoint
    - Boolean algebras: atomic, atomless
    - Groups: abelian, torsion free, divisible, nilpotent, solvable
  - There are invariant properties of models that are not representable by first-order sentences
    - Partial orders: any two elements are connected by a finite chain

- Peano arithmetic: is an end extension
- Groups: has n distinct normal subgroups, finite, free, simple

15. Where can we get more insight about how to describe the invariant properties of a model M?

- The category $D(M)$ of definable subsets and definable functions.[2]
  - Fact: If $h : M \to N$ is an isomorphism, then $D(M)$ is isomorphic to $D(N)$.
  - Conjecture: If $D(M)$ is isomorphic to $D(N)$, then M is isomorphic to N.
- McGee has results that indicate that all the invariant features of M can be described in $\mathscr{L}_{\infty\infty}$.
- Vaananen has a proposal about how the signature Σ defines higher order sorts.[3]

[2] see Pillay, "Model theory" *Notices of the AMS* 2000

[3] "Sort logic and foundations of mathematics." In *Infinity and truth*

16. Revolutionary Structuralism I: Elementary theory of the category of sets (ETCS)

- Key fact: In ETCS, no two sets can share a point in common
- ETCS is a two sorted first-order theory: objects and arrows
- Axioms: initial object, terminal object, Cartesian products, coproducts, natural number object, etc.
- A **point** of X is a morphism $p : 1 \to X$
- Each isomorphism $h : X \to Y$ gives a way of identifying points in X with points in Y.

17. Mitchell's Theorem: ETCS is "equivalent" to a weak form of ZF set theory

- Open question: what does "equivalence" mean here?
  - If it's merely "equal logical strength" then it's too weak to carry interpretive weight (e.g. Gödel translation of intuitionistic logic)
  - Bi-interpretability? Definitional equivalence?
  - Morita equivalence?
- Could category-theoretic foundations could be *equivalent* in some precise sense to set-theoretic foundations?