Dispensing with the Hole Argument

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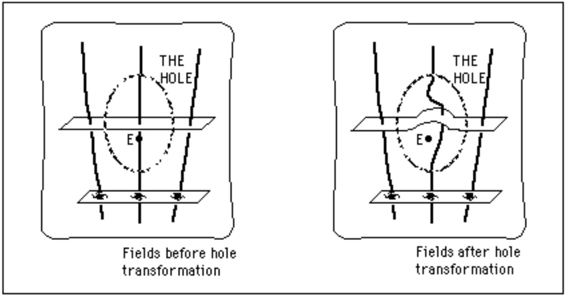
Lochbetrachtung

Einstein gets confused about how differential geometry works



General covariance?

Earman and Norton, What price spacetime substantivalism?

- Space substantivalism I (Newton): space is something sui generis that always was and always will be
- Spacetime substantivalism II: spacetime points exist
- Spacetime substantivalism III: these two models represent different possible worlds

Pulling back a metric

- A **smooth manifold** is a set M with a family of charts $\{\psi : U \to \mathbb{R}^n | U \subseteq M\}$ satisfying a compatibility requirement.
- A **diffeomorphism** is a smooth map $\varphi : M \to N$ that has a smooth inverse.
- A **spacetime** (M, g) is a smooth manifold equipped with a Lorentzian metric g.
- If $\varphi : M \to N$ is a diffeomorphism, and g is a Lorentzian metric on N, then $\varphi^* g$ is a Lorentzian metric on M.

The hole argument down to a point

- Given $\varphi : M \to M$ with $\varphi(p) \neq p$ and $g(p) \neq g(\phi(p))$
- In (M, g) we have $p \mapsto g(p)$
- In $(M, \varphi^* g)$ we have $p \mapsto g(\phi(p))$

The hole argument in a nutshell

1. Substantivalism: the models (M, g) and $(M, \varphi^* g)$ represent different possible worlds
2. Indeterminism: the models (M, g) and $(M, \varphi^* g)$ agree on everything in the past of the hole O.

Weatherall's mathematical dissolution

- Even a substantivalist should grant that (M, g) and $(M, \varphi^* g)$ represent the same possible world
- The standard for comparison of (M, g) and $(M, \varphi^* g)$ is the isomorphism φ, not the identity 1_M
  - Compare $p \mapsto g(p)$ with $\varphi(p) \mapsto (\varphi^* g)(p) = g(\varphi(p))$
  - 1_M is not an isomorphism between (M, g) and $(M, \varphi^* g)$

Metaphysics revanche

- Read and Pooley: Weatherall is assuming mathematical structuralism
- Teitel: Weatherall mistakenly assumes that mathematics solves the metaphysical question

A practical/mathematical turn

- Shouldn't the interpretation of physical theories be independent of the **niceties** of the foundations of mathematics?
  - Axioms about higher cardinal numbers
  - Perhaps even the axiom of choice
- Granted that there *are* substantive metaphysical disputes, it is surely the case that some of our questions are pseudo-questions, especially where difficult mathematics is involved
- Goal: clear conceptual reasoning about highly mathematized theories
- Goal: understanding how to represent theoretical commitment through the use of mathematical formalism
- Goal: deciding whether physical theories and/or the foundations of mathematics need to be reformed

How to reason with models

- **Democracy:** When reasoning with models, there is no default standard of cross-model identity of points.
- Equivalently: A model M can always be replaced by an isomorphic model M'.

How to reason with models

- **Counting:** When reasoning with models, we never need to count how many models are in an isomorphism class
- Set-theoretic foundations (i.e. model theory based on ZF set theory) invites violations of Monads and Counting
- WARNING: Counting does not imply that isomorphic models are identical

The paradox of symmetry

1. If there is an isomorphism $\varphi : M \to N$ then $M = N$.
2. There are true counterfactual statements.
   - "X could have had a different velocity than it actually has"
   - "More than one velocity is possible"
   - "If a force were applied to X, then it would change velocity"

This paradox is especially acute in theories with transitive symmetry group, e.g. Galilean relativity, special relativity, quantum mechanics.

The categorical mindset

- A **category** C consists of objects C_0 and arrows C_1
- Category of models: take C_0 to be models and C_1 to be symmetries
- A **functor** $F : C \to D$ consists of a pair of maps, one on objects and one on arrows
- Categories C and D are **equivalent** just in case there are functors $F : C \to D$ and $G : D \to C$ that are quasi-inverse
- Fact: An equivalence of categories induces a bijection of isomorphism classes of objects
- Fact: Equivalent categories can nonetheless have different numbers of objects in the equivalence classes

Manchak and Halvorson

- Claim: There are no non-trivial hole-isometries
- Geroch's Theorem: If $\varphi, \psi : (M, g) \to (N, h)$ are isometries that agree on an open subset, then they agree everywhere
- Corollary: If $\varphi : (M, g) \to (M, g)$ doesn't change anything outside of O, then it doesn't change anything inside O
- Is the diffeomorphism $\varphi : M \to M$ that forms the basis for the hole argument trivial or not?

Key questions

- Are the issues here unique to diffeomorphisms (general covariance), or are we faced with a general issue about to reason with models?
  - Compare $\langle\{a, b\}, \{a\}\rangle$ with $\langle\{a, b\}, \{b\}\rangle$
- Diffeomorphism versus isomorphism: are the former even relevant?
- Is Weatherall's response about correct reasoning, or does it assume mathematical structuralism?
  - Does the hole argument contain tacit premises about mathematical ontology?

Elementary Theory of the Category of Sets (ETCS)

- Two sorted first-order theory: Objects and Arrows
- Axioms: initial object, terminal object, Cartesian products, coproducts, natural number object, etc.
- A **point** of X is a morphism $p : 1 \to X$
- By definition, two sets X and Y cannot share a point in common. (The equality symbol is not defined across different sets.)
- But each isomorphism $h : X \to Y$ gives a way of identifying points in X with points in Y.

ETCS as a foundation for mathematical physics?

- Can the reasoning of physicists be reproduced without assuming a default standard of sameness of points across models? i.e. do models have to be based on the same "ur domain"?
- Example: There is a model just like M, except that the point a is not in the extension of P.
- Real examples??

Mitchell's Theorem

- ETCS is "equivalent" to a weak form of ZF set theory
- Open question: what does "equivalence" mean here?
  - If it's merely "equal logical strength" then it's too weak to carry interpretive weight (compare Gödel translation of intuitionistic logic)
  - Bi-interpretability?
  - Morita equivalence?
- Open question: does this equivalence support the two model-theoretic principles (opacity and/or vagueness)?
- It is independently interesting whether category-theoretic foundations could be *equivalent* in some precise sense to set-theoretic foundations

From ETCS to Topos Theory

Fact: ETCS axioms are a special case of axioms for a Boolean topos

(Homotopy) Type Theory

- Background
- Univalence axiom: isomorphic objects are identical
- Idiot-proof
- Philosophers have argued that UF dissolves the hole problem: Dougherty, Ladyman and Presnell
- Once again: Is there a substantive choice between ZF, ETCS (topos theory), UF? Could it make a difference for mathematical physics in practice?

Summary and Way Forward

- The hole argument has gone meta
  - Most parties are no longer concerned with the tenability of substantivalism
  - Is the hole issue just a confusion, or do different philosophical commitments lead to different ways of reasoning with models?
  - How does foundations of math figure into all of this?
- There are open technical questions
  - Equivalence of ETCS and weak ZF

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