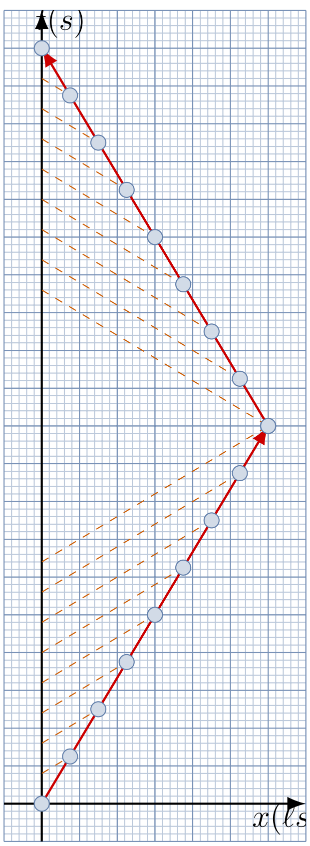
Lecture 21: Twins paradox

The scenario:

1. A and B meet
2. A and B part ways
3. Time passes (for both A and B)
4. A turns around
5. Time passes (for both A and B)
6. A and B meet again
7. Less time has passed for A than has passed for B



1. Clarifications

   (a) The last claim is a *prediction* of STR

   (b) According to the standard interpretation of STR, both A and B have kept time correctly

2. Is this a *paradox*?

   If a theory predicts P and not-P, then it is *bad*

3. How does STR predict this?

   Some terminology:

   - World line = path through spacetime traced by a massive object
   - Proper time = time elapsed along a world line

   The *clock hypothesis* states that an ideal clock measures the *distance* along a world line

   We want to define a notion of "distance" between points $p, q \in M$. The best way to do this is indirectly, via talking about the vectors at the points p and q

   Assumption. The space T_p is isomorphic to the space T_q for all $p, q \in M$. We use the name V for this space.

   Definition. A *vector space* V over the real numbers $\mathbb{R}$ has a distinguished element $0 \in V$ and two operations:

   - Addition: $v, w \mapsto v + w$
   - Scalar multiplication: $a, v \mapsto av$

   Definition. An *inner product* η on V is a function from pairs of elements of V to $\mathbb{R}$ that is:

   - Symmetric
   - Linear in both arguments
   - Semi-definite: for each $u \in V$ there is a $v \in V$ such that $\eta(u, v) \neq 0$.

   Definition. A *subspace* W of V is a subset that contains 0 and is closed under the addition and scalar multiplication operations.

   Definition. We say that η is positive definite on W just in case $\eta(v, v) \geq 0$ for all $v \in W$, and $\eta(v, v) = 0$ only if $v = 0$.

   Definition. The *signature* of η is a pair of non-negative integers (n^+, n^-), where n^+ is the maximal possible dimension for a positive definite subspace, and n^- is the maximal possible dimension for a negative definite subspace.

Assumption. Minkowski spacetime is a metric affine space with signature $(1, 3)$, although we often represent it by an affine space with signature $(1, 1)$.

Definition. For $u \in V$, we let $\|u\| = |\langle u, u\rangle|^2$, which we consider to be the generalized length of u.

Proposition (Reverse triangle inequality)**.** Let $u, v \in V$ be timelike vectors. Then $\|u + v\| \geq \|u\| + \|v\|$, with equality iff $u = av$ for some $a \in \mathbb{R}$.

4. What, if anything is the takeaway point of this example?

    (a) Is there no objective notion of the amount of time between events? Whose time is real?

    "You have two hours to complete and submit the exam"

5. Confusions

    (a) The physical situations of A and B are exactly the same

    (b) The difference is explained by the fact that A accelerates and B does not

    (c) The faster a clock moves the slower it ticks

6. Clarifications for the technically interested

    (a) The world lines of A and B are *not* symmetric. "α is straight" is definable in the language of STR. Since α is bent and β is straight, they are not equivalent.

    (b) In fact, spacetime distance is invariant under symmetries. Hence, it is impossible to map a worldline α to a longer worldline β. (Paradox avoided: two observers will always agree about comparisons of proper times.)

    Contrast with two observers' judgments of whether x occurred before y, when x and y are spacelike related