Binary relations on a three-element set

Let $A = \{a, b, c\}$. A *binary relation* R on A is any subset of $A \times A$. Equivalently, R is determined by its 3×3 adjacency matrix $(r_{xy})_{x,y \in A}$, where $r_{xy} = 1$ iff Rxy. Since $|A \times A| = 9$, each ordered pair may or may not belong to R. Hence the total number of binary relations on A is $2^9 = \mathbf{512}$.

We say that two relations R, R' are the same *type* if one can be obtained from the other by permuting the elements of A; i.e., there is some $\sigma \in S_3$ with

$$\langle x, y \rangle \in R \iff \langle \sigma x, \sigma y \rangle \in R'.$$

The group S_3 acts on the set of all relations in this way. Burnside's lemma gives

$$\frac{1}{6}\left(2^9 + 3\cdot 2^5 + 2\cdot 2^3\right) = 104,$$

so there are **104** distinct relation types (isomorphism classes) under relabeling.[1]

Fact. *Let φ be a sentence with the binary relation symbol R and no names. Let M and N be interpretations of R such that R^M is of the same type as R^N. Then $M \vDash \varphi$ iff $N \vDash \varphi$.*

We now describe some of these 104 types in detail, focusing on three important families: the *equivalence relations*, the *symmetric relations*, and the *asymmetric relations*.

1 Equivalence relations

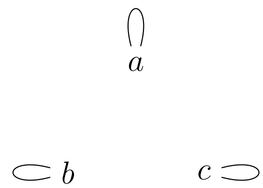
Equivalence relations correspond to partitions of A. Up to isomorphism there are precisely three partitions (hence three types).

E1: $\{\{a\}, \{b\}, \{c\}\}$ (identity relation)

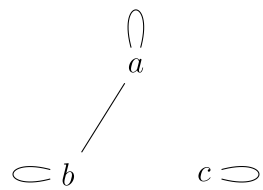
[1] **Burnside's Lemma.** Let a finite group G act on a finite set X. For each $g \in G$, let $\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$ be the set of elements fixed by g. Then the number of distinct orbits of X under the action of G is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

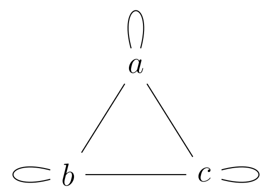
In particular, if G acts by permuting coordinates of combinatorial structures (such as the rows and columns of a matrix), the lemma gives the number of unlabeled structures up to relabeling.



E2: $\{\{a, b\}, \{c\}\}$ (one 2–class + one singleton)



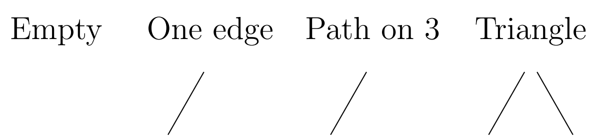
E3: $\{\{a, b, c\}\}$ (universal relation)



2 Symmetric relations (20 types)

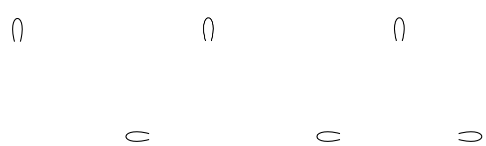
A symmetric relation is an *undirected* graph with optional loops. Up to isomorphism, start from the underlying simple graph (no loops), then add loops modulo its automorphisms.

Underlying graph shapes (no loops):

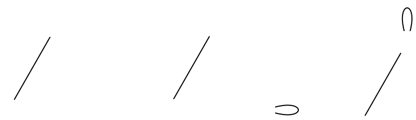


Each base shape can receive loops in various ways, counted up to automorphisms.

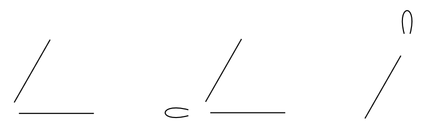
(a) Empty graph. All vertices are indistinguishable; loops can be placed on any subset. Distinct cases (by number of loops): 0, 1, 2, or 3.



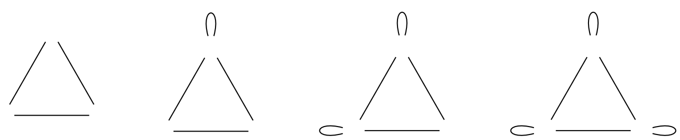
(b) One edge. Automorphism swaps the two adjacent vertices; the isolated vertex is distinct. Loops on the pair: 0, 1, or 2; loop on the isolated: 0 or 1. This gives $3 \times 2 = 6$ types.



(c) Path on three vertices. Automorphism swaps the two endpoints; the middle vertex is distinguished. Loops on endpoints: 0, 1, or 2; loop on middle: 0 or 1. Again $3 \times 2 = 6$ types.



(d) Triangle. All vertices indistinguishable; loops can be placed on any subset. Distinct cases (by number of loops): 0, 1, 2, or 3.



Count check. Distinct loop patterns modulo automorphisms:

$$4\ (\text{empty})\ +\ 6\ (\text{one edge})\ +\ 6\ (\text{path})\ +\ 4\ (\text{triangle})\ =\ \boxed{20}.$$

Thus there are **20 symmetric relation types** on a 3–element set.

Every symmetric relation on $A = \{a, b, c\}$ can be represented by a 3×3 Boolean matrix (r_{ij}) such that $r_{ij} = r_{ji}$ for all i, j. Each r_{ij} records whether the ordered pair $\langle a_i, a_j \rangle$ belongs to the relation R.

To understand how many distinct symmetric relations there are up to isomorphism, it helps to build these matrices column by column. The first column determines all entries in the first row by symmetry, so we can visualize the process as gradually filling in the upper–left triangle of the matrix.

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

The first column corresponds to the loops and connections involving the first element a. We can freely choose:

- whether a has a loop ($r_{11} = 0$ or 1), and

- whether a is connected to b (r_{12}) and to c (r_{13}).

Hence there are $2^3 = 8$ possibilities for the first column (and its symmetric first row).

Now fix one of those eight possibilities. Once the first column is chosen, some entries in the second column are already constrained by symmetry: the entry r_{21} is equal to r_{12}. Thus the second column still has two unconstrained positions: r_{22} (loop on b) and r_{23} (connection between b and c). That gives $2^2 = 4$ degrees of freedom for the second column.

Finally, once we have filled in the first and second columns, the third column is almost completely determined by symmetry: $r_{31} = r_{13}$ and $r_{32} = r_{23}$ are fixed, and only r_{33} (the loop on c) remains free.

Altogether we have

$$2^3 \text{ (for the first column)} \times 2^2 \text{ (for the second column)} \times 2^1 \text{ (for the third column)} \;=\; 2^6 = 64$$

labeled symmetric relations.

When we ignore the names of a, b, and c—that is, when we identify relations that differ only by renaming the elements of A— many of these 64 labeled matrices coincide. For example, the two matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

represent the same relation *type*: each depicts a single undirected edge connecting two of the three elements, with the remaining element isolated. They differ only in which pair of vertices happens to be connected. When we collapse all such relabelings, the 64 labeled matrices fall into exactly **20 distinct unlabeled patterns**. These correspond precisely to the 20 symmetric relation types classified earlier by their underlying graph shapes and possible loop configurations.

3 Asymmetric relations

A relation R on $A = \{a, b, c\}$ is *asymmetric* if

$$Rxy \;\Rightarrow\; \neg Ryx.$$

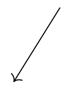
Equivalently, R has no loops and no pair of opposite arrows; in graph terms, it is a loopless digraph without 2–cycles. (Asymmetric $\Rightarrow$ irreflexive.)

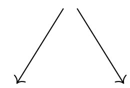
For each unordered pair $\{x, y\}$ there are exactly three options: $(x \to y)$, $(y \to x)$, or neither. With $\binom{3}{2} = 3$ pairs, the number of *labeled* asymmetric relations on A is

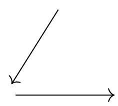
$$3^3 = 27.$$

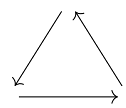
Representative shapes up to relabeling (no loops anywhere):

- **Empty relation.**

- **Single arrow** $(a \to b)$. 

- **Fork (two arrows with a common source)** $(a \to b,\ a \to c)$. 

- **Chain** $(a \to b \to c)$. 

- **3–cycle (tournament on three)** $(a \to b \to c \to a)$. 

4 Summary

- $2^9 = 512$ labeled binary relations on $A = \{a, b, c\}$.
- 104 relation types up to relabeling of the domain.
- Equivalence relations split into 3 types (partitions of A).
- Symmetric relations split into 20 types (simple-graph shape $\times$ loop–pattern modulo automorphisms).