Review: translations and proofs

PHI 201 – Introductory Logic

November 17, 2025

Proofs

Quantifiers of the same type

$$\forall x \forall y\, \varphi \;\equiv\; \forall y \forall x\, \varphi$$

$$\exists x \exists y\, \varphi \;\equiv\; \exists y \exists x\, \varphi$$

Fact: Quantifiers of the same type commute. That is, universal quantifiers commute with each other, and existential quantifiers commute with each other.

Prenex normal form

Fact: Every sentence in predicate logic is equivalent to a sentence where all the quantifiers occur at the beginning.

Pushing negation inside

$$\neg \forall x\, \varphi \equiv \exists x \neg \varphi$$

$$\neg \exists x\, \varphi \equiv \forall x \neg \varphi$$

Equivalent formulas

For purposes of establishing equivalence, you can treat the variables inside a formula as names.

If quantifiers are stripped, and variables replaced by distinct names, then the process can be reversed to put the quantifiers back on.

Equivalent formulas

Fact: If $\varphi(a_1, \dots, a_n)$ is equivalent to $\psi(a_1, \dots, a_n)$, then $Q_1 x_1 \cdots Q_n x_n\, \varphi(x_1, \dots, x_n)$ is equivalent to $Q_1 x_1 \cdots Q_n x_n\, \psi(x_1, \dots, x_n)$.

Practical upshot: you can detect equivalence by looking at the formula inside quantifiers.

1	(1)	$\forall x \exists y \forall z((Fx \land Gy) \to Hz)$	A
1	(2)	$\exists y \forall z((Fa \land Gy) \to Hz)$	1 UE
3	(3)	$\forall z((Fa \land Gb) \to Hz)$	A
3	(4)	$(Fa \land Gb) \to Hc$	3 UE
3	(5)	$(\neg Fa \lor \neg Gb) \lor Hc$	4 TTV
3	(6)	$\forall z((\neg Fa \lor \neg Gb) \lor Hz)$	5 UI
3	(7)	$\exists y \forall z((\neg Fa \lor Gy) \lor Hz)$	6 EI
1	(8)	$\exists y \forall z((\neg Fa \lor Gy) \lor Hz)$	2,3,7 EE
1	(9)	$\forall x \exists y \forall z((\neg Fx \lor Gy) \lor Hz)$	8 UI

$$
\begin{aligned}
Fy \wedge \exists x Gx &\equiv \exists x(Fy \wedge Gx) \\
Fy \vee \exists x Gx &\equiv \exists x(Fy \vee Gx) \\
Fy \rightarrow \exists x Gx &\equiv \exists x(Fy \rightarrow Gx)
\end{aligned}
$$

$$\begin{aligned} \exists x(Gx \wedge Fy) &\equiv \exists x Gx \wedge Fy \\ \exists x(Gx \vee Fy) &\equiv \exists x Gx \wedge Fy \\ \exists x(Gx \rightarrow Fy) &\equiv \forall x Gx \rightarrow Fy \end{aligned}$$

$$
\begin{aligned}
\forall y(\exists x Gx \to Fy) &\equiv \forall y \forall x(Gx \to Fy) \\
&\equiv \exists x Gx \to \forall y Gy
\end{aligned}
$$

$$\forall x Fx \to Gb \ \vdash \ \exists x(Fx \to Gb)$$

Test your understanding

Use what you just learned to quickly see that the following sentence is a tautology:

$$\forall y \exists x (Rxy \to \forall z Rzy)$$

Translation

Nested Quantifiers and Scope

Source note. The next few slides are based in part on Warren Goldfarb's *Deductive Logic*.

Nested quantifiers introduce a new complication: we must determine *which quantifier governs which.*

- Quantifiers have **scope**, and one may fall inside another.
- To paraphrase, it helps to work **from the outside in**.
- First decide whether the statement as a whole is **universal** ($\forall$) or **existential** ($\exists$).
- Then paraphrase the **open formula** inside its scope.

Example Sentences

We assume—for now—that the **universe of discourse** is the class of **persons**.

Consider the following English sentences:

1. Every critic admires some painter.
2. Every critic is admired by some painter.
3. Every critic admires all painters.

All three sentences are **universal quantifications**.

Example (1): Every critic admires some painter

Step 1 (outer quantifier):

$$\forall x\,(x \text{ is a critic} \rightarrow \dots)$$

Step 2 (inner scope):

$$x \text{ admires some painter} \quad \rightsquigarrow \quad \exists y\,(y \text{ is a painter} \land x \text{ admir}$$

Final formalization:

$$\forall x\,(Cx \rightarrow \exists y\,(Py \land Axy))$$

Example (2): Every critic is admired by some painter

Paraphrase structure:

$$\forall x\,(Cx \to \exists y\,(Py \land Ayx))$$

Note the reversal in the predicate:

$$Ayx \equiv y \text{ admires } x.$$

Final symbolic form:

$$\forall x\,(Cx \to \exists y\,(Py \land Ayx))$$

Example (3): Every critic admires all painters

Step 1 (outer quantifier):

$$\forall x\,(Cx \to \ldots)$$

Step 2 (scope):

$$x \text{ admires all painters} \quad \rightsquigarrow \quad \forall y(Py \to Axy)$$

Final formalization:

$$\forall x\,(Cx \to \forall y(Py \to Axy))$$

Existential sentences

There is a painter who is admired by every critic.

$\exists x(x$ is a painter $\wedge\, x$ is admired by every critic$)$

Some critics admire all painters.

There is a critic who admires no painters.

P's bear the relation R only to Q's

$$\forall x(Px \to \forall y(Rxy \to Qy))$$

Conversational implicature of if and only if: "Danes only trust other Danes."

$$\forall x(Dx \to \forall y(Txy \leftrightarrow Dy))$$

Only the F that/who are G are H.

$$\forall x(Fx \rightarrow (Hx \rightarrow Gx))$$

$$\forall x(Hx \rightarrow (Fx \wedge Gx))$$

P's bear the relation R only to Q's that/who are S.

a only respects Harvard professors who acknowledge that Princeton is superior.

$$\forall x((Hx \land Rax) \rightarrow Ax)$$

$$\forall x(Rax \rightarrow (Hx \land Ax))$$