

Theories

PHI 201 — Introductory Logic

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The utopian vision of symbolic logic

- Two original hopes for symbolic logic.
 - ① It provides a universal language for science.
 - ② It dissolves philosophical pseudo-problems.
- While this doesn't work out so easily in practice, there is a sense in which all “theories” in mathematics can be formalized in predicate/relational logic.

Equality

Equality is a special relation

- Equality is a binary relation which we write as an infix rather than as a prefix

$$c = d, \exists x(x = d), \forall y \exists x(x = y)$$

- Using “=” allows us to express many new things.

At least n

$$\exists x \exists y (x \neq y)$$

$$\exists x \exists y \exists z ((x \neq y \wedge x \neq z) \wedge y \neq z)$$

At most n

$$\forall x \forall y (x = y)$$

$$\forall x \forall y \forall z ((x = y \vee x = z) \vee y = z)$$

Exactly n

$$\exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$$

$$\begin{aligned} \exists x \exists y \exists z & (((x \neq y \wedge x \neq z) \wedge y \neq z) \\ & \wedge \forall w ((w = x \vee w = y) \vee w = z)) \end{aligned}$$

There is a unique P

$$\exists x(Px \wedge \forall y(Py \rightarrow x = y))$$

Definite descriptions

Superlatives

“There is a tallest student.”

$$\exists x \forall y (x \neq y \rightarrow Txy)$$

This sentence entails uniqueness only because we implicitly assume that “taller than” is asymmetric.

$$\forall x \forall y (Txy \rightarrow \neg Tyx)$$

1	(1)	$\exists x \forall y (x \neq y \rightarrow Txy)$	A
2	(2)	$\forall y (a \neq y \rightarrow Tay)$	A
3	(3)	$\forall y (b \neq y \rightarrow Tby)$	A
4	(4)	$a \neq b$	A
2	(5)	$a \neq b \rightarrow Tab$	2 UE
3	(6)	$b \neq a \rightarrow Tba$	3 UE

Inference rules for equality

$\Gamma \quad (m) \quad \varphi(a)$

$\Delta \quad (n) \quad a = b$

$\Gamma, \Delta \quad (o) \quad \varphi(b) \qquad \qquad m, n = E$

To show: $a = b, b = c \vdash a = c$

$$1 \quad (1) \quad a = b$$

A

$$2 \quad (2) \quad b = c$$

A

$$1,2 \quad (3) \quad a = c$$

2,1 =E

Inference rules for equality

$$(m) \quad a = a \qquad \qquad \qquad =I$$

To show: $a = b \vdash b = a$

$$1 \quad (1) \quad a = b$$

A

$$(2) \quad a = a$$

=I

$$1 \quad (3) \quad b = a$$

2,1 =E

Nobody but

Alice respects nobody but Bob.

$$Rab \wedge \forall x(Rax \rightarrow x = b)$$

$$\forall x(Rax \leftrightarrow x = b)$$

Everybody loves my baby

1	(1)	$\forall x Lxb$	A
2	(2)	$\forall y (Lby \rightarrow y = a)$	A
1	(3)	Lbb	1 UE
2	(4)	$Lbb \rightarrow b = a$	2 UE

- The theory of equality is peculiar, because we build its axioms in as new inference rules.
- Now we look at theories whose axioms are sentences.

Theory of partial order

transitive:

$$\forall x \forall y \forall z ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$$

reflexive:

$$\forall x (x \leq x)$$

antisymmetric:

$$\forall x \forall y ((x \leq y \wedge y \leq x) \rightarrow x = y)$$

A cornucopia of partially ordered sets

linear:

$$\forall x \forall y ((x \leq y) \vee (y \leq x))$$

- What's a sentence that is true of the natural numbers $1, 2, 3, \dots$ but false of the integers $\dots, -2, -1, 0, 1, 2, \dots ?$
- What's a sentence that is true of the integers but false of the rational numbers?

Set theory

extensionality

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

existence of an emptyset

$$\exists z \forall x (x \notin z)$$

Uniqueness of the emptyset

1 (1) $\forall x(x \notin a)$

A

2 (2) $\forall x(x \notin b)$

A

1 (3) $c \notin a$

1 UE

1 (4) $c \in a \rightarrow c \in b$

3 neg par

Naive set theory

comprehension

$$\exists x \forall y (y \in x \leftrightarrow \varphi(y))$$

Consistent theories

We say that a theory T is **consistent** if there is no sentence φ such that both $T \vdash \varphi$ and $T \vdash \neg\varphi$.

Naive set theory is inconsistent

Use comprehension with the predicate “ $y \notin y$ ”

$$1 \quad (1) \quad \exists x \forall y (y \in x \leftrightarrow y \notin y) \quad A$$

$$2 \quad (2) \quad \forall y (y \in a \leftrightarrow y \notin y) \quad A$$

$$2 \quad (3) \quad a \in a \leftrightarrow a \notin a \quad 2 \text{ UE}$$

Sophisticating set theory

pairing

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

separation: For every formula $\varphi(x, b_1, \dots, b_n)$,

$$\forall y \exists z \forall x (x \in z \leftrightarrow (x \in y \wedge \varphi(x, b_1, \dots, b_n))).$$

Existence and uniqueness of intersections

	(1)	$\exists z \forall x (x \in z \leftrightarrow (x \in a \wedge x \in b))$	sep
2	(2)	$\forall x (x \in c \leftrightarrow (x \in a \wedge x \in b))$	A
	(3)	$\forall y \forall y' (\forall x (x \in y \leftrightarrow x \in y') \rightarrow y = y')$	ext
4	(4)	$\forall x (x \in d \leftrightarrow (x \in a \wedge x \in b))$	A
2,4	(5)	$\forall x (x \in c \leftrightarrow x \in d)$	2,4
	(6)	$\forall x (x \in c \leftrightarrow x \in d) \rightarrow c = d$	3 UE
2,4	(8)	$c = d$	6,5 MP