

# Lecture 7: Invitation to Predicate Logic

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# Overview

- 1 Motivation: Propositional logic cannot see all logical relations
- 2 A more fine-grained grammar
  - Names and predicates
  - Variables and quantifiers
- 3 Translation
- 4 Inference rules
  - $\forall$  elimination
  - $\forall$  introduction

Propositional logic is inadequate

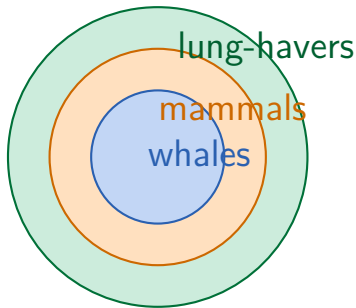
# Validities that escape propositional logic

- All people are mortal.
- Socrates is a person.
- Therefore, Socrates is mortal.

If the subject and predicate of sentences are not both the same, then propositional logic does not recognize any relation between them.

# Validities that escape propositional logic

- All **whales** are **mammals**.
- All **mammals** have **lungs**.
- Therefore: All **whales** have **lungs**.



$$\text{All}(W, M), \text{All}(M, L) \vdash \text{All}(W, L)$$

# When propositional logic falls short

## Example from mathematics

If a number is even, its square is even. 4 is even.  $\therefore 4^2$  is even.

**Propositional view:**

$$P, P \rightarrow Q \vdash Q$$

**Mathematical structure:**

$$\forall n (E(n) \rightarrow E(n^2)), E(4) \vdash E(4^2)$$

# Diagnosis

- The inadequacy of propositional logic cannot be fixed by adding more inference rules.
  - If we add any additional rules, then our system would become inconsistent.
- Have we missed some propositional connectives?
  - No, there is a precise sense in which our set of connectives is **conceptually complete**.

# The predicate calculus

- In the early 20th century, the missing logical structure was identified, represented symbolically, and codified in a “calculus”.



# Sub-propositional grammar

# Names and predicates

Alice is French.

Bernard is French.

Alice is German.

# Quantified sentences

- You are familiar with the concept of a variable from mathematics.
- Natural languages do not explicitly use variables.
- Hypothesis: “All” and “Some” sentences are best analyzed as consisting of predicate symbols, variables, and quantifiers.

# Variables

Alice is French.

$Fa$

$x$  is French.

$Fx$

Someone is French.

?

There is an  $x$  such that  $x$  is French.

$\exists x Fx$

# Formulas

- We don't call " $Fx$ " a proposition, since it cannot be true or false.
- We call " $Fx$ " a **formula**.
- Adding the quantifier " $\exists x$ " to " $Fx$ " creates a sentence.

# Universal quantifier

All whales are mammals.

?

If  $a$  is a whale then  $a$  is a mammal.

$Wa \rightarrow Ma$

For any  $x$ , if  $x$  is a whale then  $x$  is a mammal.

$\forall x(Wx \rightarrow Mx)$

# Standard syllogistic forms

**All Finns are gregarious.**

$$\forall x (Fx \rightarrow Gx)$$

**Some Finns are gregarious.**

$$\exists x (Fx \wedge Gx)$$

**No Finns are gregarious.**

$$\forall x (Fx \rightarrow \neg Gx)$$

**Some Finns are not gregarious.**

$$\exists x (Fx \wedge \neg Gx)$$

All happy Finns are gregarious.

All Finns and Germans are happy.



No dogs or cats are permitted in the restaurant.

$$\forall x(Fx \rightarrow P)$$

Everything has the feature that if it is  $F$ , then  $P$  holds.

$$\forall xFx \rightarrow P$$

If everything has the feature  $F$ , then  $P$  holds.

# Relations

Maren is taller than Niels.

Maren is taller than someone.

Someone is taller than Niels.

Everyone is taller than someone.

Someone is taller than everyone.

There is a student who admires every professor.

$$\exists x(Sx \wedge \forall y(Py \rightarrow Axy))$$

There is a professor whom every student admires.

$$\exists x(Px \wedge \forall y(Sy \rightarrow Ayx))$$

Every student admires some professor.

$$\forall x(Sx \rightarrow \exists y(Py \wedge Axy))$$

# Inference to/from quantified statements

## $\forall$ elimination

The idea behind  $\forall$  elimination is straightforward:

From a universal statement, any **instance** follows logically.

$$\frac{\forall x \varphi(x)}{\varphi(a)}$$

## $\forall$ elimination

$\forall x(Fx \rightarrow Gx), Fa \vdash Ga$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$Fa$	A
1	(3)	$Fa \rightarrow Ga$	1 UE
1,2	(4)	$Ga$	3,2 MP



$$\forall x \forall y (Fx \wedge Gy) \vdash Fa \wedge Gb$$

1	(1)	$\forall x \forall y (Fx \wedge Gy)$	A
1	(2)	$\forall y (Fa \wedge Gy)$	1 UE
1	(3)	$Fa \wedge Gb$	2 UE

$$\forall x \forall y (Fx \wedge Gy) \vdash Fa \wedge Ga$$

1	(1)	$\forall x \forall y (Fx \wedge Gy)$	A
1	(2)	$\forall y (Fa \wedge Gy)$	1 UE
1	(3)	$Fa \wedge Ga$	2 UE

$$P \rightarrow \forall x Fx \vdash P \rightarrow Fa$$

1	(1)	$P \rightarrow \forall x Fx$	A
2	(2)	$P$	A
1,2	(3)	$\forall x Fx$	1,2 MP
1,2	(4)	$Fa$	3 UE
1	(5)	$P \rightarrow Fa$	2,4 CP

$\neg Fa \vdash \neg \forall x Fx$

1	(1)	$\neg Fa$	A
2	(2)	$\forall x Fx$	A
2	(3)	$Fa$	2 UE
1,2	(4)	$Fa \wedge \neg Fa$	3,1 $\wedge$ I
1	(5)	$\neg \forall x Fx$	2,4 RA

# Warnings

Only apply UE when the entire sentence on the line is universally quantified.

$$\forall x(Fx \rightarrow P)$$

$$\forall x(Fx \rightarrow \forall yGy)$$

$$\forall xFx \rightarrow Ga$$

$$\forall x\forall yRxy$$

# Warnings

When applying UE, replace all instances of the relevant variable with the same name.

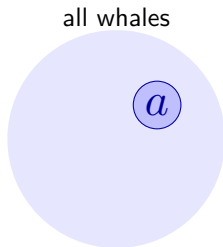
$$\forall x(Fx \rightarrow \forall yRxy)$$

# From one individual to everyone

## Intuitive idea

To show that **everyone** has a property, we can reason about **one individual chosen at random**.

- Suppose we want to prove that all whales have lungs.
- We pick a whale—call it  $a$ .
- We reason about  $a$  as if it were any whale.



# What does it mean for $a$ to be *arbitrary*?

## Arbitrary name

The name  $a$  is **arbitrary** when nothing in the proof depends on any *special feature* of  $a$ .

- Our reasoning about  $a$  must not rely on facts like “ $a$  lives in the Pacific” or “ $a$  is the largest whale.”
- The argument must hold no matter which whale we picked.

*An arbitrary name stands for an individual we reason about generally*



# From arbitrariness to universal generalization

## Bridge to Universal Introduction

If we can prove  $\varphi(a)$  using  $a$  as an **arbitrary name**, then we may infer the general statement  $\forall x \varphi(x)$ .

$$\frac{\varphi(a)}{\forall x \varphi(x)} \text{UI} \quad (\text{side condition: } a \text{ not free in any open assumption})$$

- The conclusion applies to *all* objects of that kind.
- The key is that  $a$  never referred to anything special.

# Universal introduction

From a line

$$\Gamma \quad (m) \quad \varphi(a)$$

we may infer

$$\Gamma \quad (n) \quad \forall x \varphi(x)$$

provided that the name “ $a$ ” does not occur in any of the sentences listed in  $\Gamma$  or in  $\varphi(x)$ .

# $\forall$ introduction

$$\forall x(Fx \rightarrow Gx), \forall xFx \vdash \forall xGx$$

1	(1)	$\forall x(Fx \rightarrow Gx)$	A
2	(2)	$\forall xFx$	A
2	(3)	$Fa$	2 UE
1	(4)	$Fa \rightarrow Ga$	1 UE
1,2	(5)	$Ga$	4,3 MP
1,2	(6)	$\forall xGx$	5 UI

$\vdash \forall x(Fx \rightarrow (Fx \vee Gx))$

1	(1)	$Fa$	A
1	(2)	$Fa \vee Ga$	1 $\vee$ I
$\emptyset$	(3)	$Fa \rightarrow (Fa \vee Ga)$	1,2 CP
$\emptyset$	(4)	$\forall x(Fx \rightarrow (Fx \vee Gx))$	3 UI

$$\forall x(P \rightarrow Fx) \vdash P \rightarrow \forall xFx$$

1	(1)	$\forall x(P \rightarrow Fx)$	A
2	(2)	$P$	A
1	(3)	$P \rightarrow Fa$	1 UE
1,2	(4)	$Fa$	3,2 MP
1,2	(5)	$\forall xFx$	4 UI
1	(6)	$P \rightarrow \forall xFx$	2,5 CP

$$P \rightarrow \forall x Fx \vdash \forall x (P \rightarrow Fx)$$

1	(1)	$P \rightarrow \forall x Fx$	A
2	(2)	$P$	A
1,2	(3)	$\forall x Fx$	1,2 MP
1,2	(4)	$Fa$	3 UE
1	(5)	$P \rightarrow Fa$	2,4 CP
1	(6)	$\forall x (P \rightarrow Fx)$	5 UI

# Precisifying the UI rule

$\forall$ I requires replacing **all** instances of the arbitrary name.

1	(1)	$\forall x Rxx$	A	
1	(2)	$Raa$	1 UE	
1	(3)	$\forall x Rxa$	2 UI	error!
1	(4)	$\forall y \forall x Rxy$	3 UI	

# Precisifying the UE rule

But UE does allow instantiating to a name that already occurs in the formula.

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	$Raa$	2 UE
1	(4)	$\forall x Rxx$	3 UI



# Precisifying the UE rule

UE allows us to choose any name — same or different from what already occurs.

1	(1)	$\forall x \forall y Rxy$	A
1	(2)	$\forall y Ray$	1 UE
1	(3)	$Rab$	2 UE
1	(4)	$\forall x Rxb$	3 UI
1	(5)	$\forall y \forall x Rxy$	4 UI