

# Week 3

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# Reductio ad Absurdum

# Introduction

- Idea behind Reductio ad Absurdum: Show that something is **not** the case ( $\neg A$ ) by showing that it ( $A$ ) leads, via logically valid reasoning, to a contradiction.
  - RA is truly powerful if combined with DN-elimination to establish **positive** conclusions.

# $\sqrt{2}$ is not a rational number

**Proof.** Assume for reductio ad absurdum that  $\sqrt{2}$  is rational, i.e. that  $\sqrt{2} = \frac{a}{b}$  with integers  $a, b$  in lowest terms ( $\gcd(a, b) = 1, b \neq 0$ ). Then

$$2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2.$$

Hence  $a^2$  is even, so  $a$  is even; write  $a = 2k$ . Substituting,

$$(2k)^2 = 2b^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow b^2 = 2k^2,$$

so  $b^2$  is even and therefore  $b$  is even.

Thus both  $a$  and  $b$  are even, contradicting that  $\frac{a}{b}$  is in lowest terms. Therefore,  $\sqrt{2}$  is irrational.  $\square$

# Reductio ad Absurdum

$m$	$(m)$	$A$	$A$
	$\vdots$		
$n_1, \dots, n_j$	$(n)$	$B \wedge \neg B$	
	$\vdots$		
$n_1, \dots, \widehat{m}, \dots, n_j$	$(k)$	$\neg A$	$m, n \text{ RA}$

# Reductio ad Absurdum

$$A_1, \dots, A_n, B \vdash \perp$$

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$$A_1, \dots, A_n \vdash \neg B$$

1	(1)	$\neg P \rightarrow P$	A
2	(2)	$\neg P$	A
1,2	(3)	$P$	1,2 MP
1,2	(4)	$P \wedge \neg P$	3,2 $\wedge$ I
1	(5)	$\neg\neg P$	2,4 RA
1	(6)	$P$	5 DN

# DeMorgan's laws

Show  $\neg(P \vee Q) \vdash \neg P$

1	(1)	$\neg(P \vee Q)$	A
2	(2)	$P$	A
2	(3)	$P \vee Q$	2 $\vee$ I
1,2	(4)	$(P \vee Q) \wedge \neg(P \vee Q)$	3,1 $\wedge$ I
1	(5)	$\neg P$	2,4 RA



# Material conditional

Show  $\neg(\neg P \vee Q) \vdash \neg(P \rightarrow Q)$

1	(1)	$\neg(\neg P \vee Q)$	A
2	(2)	$P \rightarrow Q$	A
1	(3)	$\neg\neg P$	see previous proof
1	(4)	$P$	3 DN
1,2	(5)	$Q$	2,4 MP
1,2	(6)	$\neg P \vee Q$	5 $\vee$ I
1,2	(7)	$(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	6,1 $\wedge$ I
1	(8)	$\neg(P \rightarrow Q)$	2,7 RA

# Law of Non-Contradiction

1	(1)	$P \wedge \neg P$	A
	(2)	$\neg(P \wedge \neg P)$	1,1 RA

# Ex Falso Quodlibet (EFQ)

1	(1)	$P$	A
2	(2)	$\neg P$	A
3	(3)	$\neg Q$	A
1,2	(4)	$P \wedge \neg P$	1,2 $\wedge I$
1,2	(5)	$\neg \neg Q$	3,4 RA
1,2	(6)	$Q$	5 DN

It is **not** required that the assumption occurs in the dependencies of the contradiction.

# Disjunctive Syllogism

$$P \vee Q, \neg P \vdash Q$$

1	(1)	$P \vee Q$	A
2	(2)	$\neg P$	A
3	(3)	$P$	A
2,3	(4)	$Q$	EFQ
5	(5)	$Q$	A
1,2	(6)	$Q$	1,3,4,5,5 $\vee$ E

# DeMorgan's Laws

$$\neg P \vee \neg Q \vdash \neg(P \wedge Q)$$

1	(1)	$\neg P$	A
2	(2)	$P \wedge Q$	A
2	(3)	$P$	2 $\wedge$ E
1,2	(4)	$P \wedge \neg P$	3,1 $\wedge$ I
1	(5)	$\neg(P \wedge Q)$	2,4 RA

# DeMorgan's Laws

$$\neg P, \neg Q \vdash \neg(P \vee Q)$$

By DS we have  $\neg P, P \vee Q \vdash Q$ .

It follows that  $\neg P, P \vee Q, \neg Q \vdash \perp$ .

By RA,  $\neg P, \neg Q \vdash \neg(P \vee Q)$ .

1	(1)	$P \vee Q$	A
2	(2)	$\neg P$	A
3	(3)	$P$	A
4	(4)	$\neg Q$	A
2,3	(5)	$P \wedge \neg P$	3,2 $\wedge I$
2,3	(6)	$\neg \neg Q$	4,5 RA
2,3	(7)	$Q$	6 DN
8	(8)	$Q$	A
1,2	(9)	$Q$	1,3,7,8,8 $\vee E$
1,2,4	(10)	$Q \wedge \neg Q$	9,4 $\wedge I$
2,4	(11)	$\neg(P \vee Q)$	1,10 RA

# Law of Excluded Middle

1	(1)	$\neg(P \vee \neg P)$	A
2	(2)	$P$	A
2	(3)	$P \vee \neg P$	2 $\vee I$
1,2	(4)	$(P \vee \neg P) \wedge \neg(P \vee \neg P)$	3,1 $\wedge I$
1	(5)	$\neg P$	2,4 RA
1	(6)	$P \vee \neg P$	5 $\vee I$
1	(7)	$(P \vee \neg P) \wedge \neg(P \vee \neg P)$	6,1 $\wedge I$
	(8)	$\neg\neg(P \vee \neg P)$	1,7 RA
	(9)	$P \vee \neg P$	8 DN



# More difficult proofs

To show:  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$

- Strategy 1: Assume negation of conclusion, apply DeMorgans. The result is two negated conditionals, which are equivalent to conjunctions.
- Strategy 2: Derive  $P \vee \neg P$ , then argue by cases. Recall that  $\neg P \vdash P \rightarrow Q$ .

# Useful sequents

**Commutativity:**  $A \wedge B \dashv\vdash B \wedge A$   
 $A \vee B \dashv\vdash B \vee A$

**Associativity:**  $(A \wedge B) \wedge C \dashv\vdash A \wedge (B \wedge C)$   
 $(A \vee B) \vee C \dashv\vdash A \vee (B \vee C)$

**Distributivity:**  $A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C)$   
 $A \vee (B \wedge C) \dashv\vdash (A \vee B) \wedge (A \vee C)$

**De Morgan's I:**  $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$   
 $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$

# Useful sequents

**Material Conditional:**  $A \rightarrow B \dashv\vdash \neg A \vee B$   
 $\neg(A \rightarrow B) \dashv\vdash A \wedge \neg B$

**Excluded Middle:**  $\vdash A \vee \neg A$

**Disjunctive Syllogism:**  $A \vee B, \neg A \vdash B$

# Truth tables

# How do you know if something can be proven?

- If you prove  $A_1, \dots, A_n \vdash B$ , then that argument form is truth preserving (in the sense that we are about to make precise).
- If you fail to prove  $A_1, \dots, A_n \vdash B$ , that doesn't prove that it is not provable.
- If you can show that  $A_1, \dots, A_n \vdash B$  is not truth-preserving, then there cannot possibly be a proof of  $A_1, \dots, A_n \vdash B$ .

# Semantic validity

- An argument form is **semantically invalid** if there is an instance of that form where the premises are true and the conclusion is false.
  - A **counterexample** to the validity of an argument is an assignment of truth values to the atomic sentences that makes that argument's premises true and its conclusion false.
- We write  $A_1, \dots, A_n \models B$  to indicate that the argument from  $A_1, \dots, A_n$  to  $B$  is semantically valid.

# Ways Things Could Be

$P$	$Q$	$R$
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

# Truth Tables

## Conjunction $\wedge$

$P$	$Q$	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

## Disjunction $\vee$

$P$	$Q$	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

## Negation $\neg$

$P$	$\neg P$
1	0
0	1

## Conditional $\rightarrow$

$P$	$Q$	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1



# Detailed truth table for $(P \wedge \neg Q) \rightarrow R$

$P$	$Q$	$R$	$( P \wedge \neg Q ) \rightarrow R$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

This sentence is a **contingency**: true in some scenarios and false in other scenarios

# Material conditional

$P$	$Q$	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

“If the Germans won World War II then French is the official language of instruction at Princeton.”

# Negative paradox is valid

$P$	$Q$	$\neg P$	$P \rightarrow Q$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1

In every case where the premise  $\neg P$  is true, the conclusion  $P \rightarrow Q$  is also true.

# Affirming the consequent is invalid

$$P \rightarrow Q, Q \not\models P$$

$P$	$Q$	$P \rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

In row 3, both premises ( $P \rightarrow Q$  and  $Q$ ) are true, but the conclusion  $P$  is false. Therefore the argument form is **invalid**.

## Ex Falso Quodlibet: $P, \neg P \therefore Q$

$P$	$Q$	$\neg P$	Premises all true?	Conclusion $Q$
1	1	0	no	1
1	0	0	no	0
0	1	1	no	1
0	0	1	no	0

The premises  $P$  and  $\neg P$  can never both be true. So there is no row where all premises are true and the conclusion false. Hence the argument form is **valid**.

# Using truth tables to guide proofs

Is there a correctly written proof with line fragments like this?

1	(1)	$P \vee Q$	A
	$\vdots$		
1	(n)	$P$	

Is there a correctly written proof with line fragments like this?

1	(1)	$P \vee Q$	A
	$\vdots$		
1	(n)	$P$	

No there cannot be. Our proof rules are **sound**, so they cannot prove a line that is semantically invalid.



# Soundness

**Fact:** If there is a correctly written proof that ends with  $A_1, \dots, A_n \vdash B$ , then  $A_1, \dots, A_n \models B$ .

Consequently, if  $A_1, \dots, A_n \not\models B$ , then there cannot be a correctly written proof that ends with  $A_1, \dots, A_n \vdash B$ .

In other words, if there is a **counterexample**, then there is no proof.

Is there a correctly written proof with line fragments like this?

1	(1)	$P \rightarrow (Q \vee R)$	$A$
	$\vdots$		
1	(n)	$(P \rightarrow Q) \vee (P \rightarrow R)$	

# Completeness

**Fact:** If  $A_1, \dots, A_n \models B$ , then the sequent  $A_1, \dots, A_n \vdash B$  can be proven.

In other words: if the argument is truth-preserving, then there is a proof.

# Semantic reasoning towards proof

We show that  $P \rightarrow (Q \vee R) \models (P \rightarrow Q) \vee (P \rightarrow R)$ .

Consider a row in the truth table where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false.

Both  $P \rightarrow Q$  and  $P \rightarrow R$  are false on this row.

$P$  is true on this row while both  $Q$  and  $R$  are false on this row.

But then  $P \rightarrow (Q \vee R)$  is false on this row.

Therefore, in every row where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false,  $P \rightarrow (Q \vee R)$  is also false.

# From informal to formal

We show that  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$ .

Consider a row in the truth table where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false.

Both  $P \rightarrow Q$  and  $P \rightarrow R$  are false on this row.

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But then  $P \rightarrow (Q \vee R)$  is false on this row.

Therefore, in every row where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false,  $P \rightarrow (Q \vee R)$  is also false.

# From informal to formal

We show that  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$ .

Assume  $\neg((P \rightarrow Q) \vee (P \rightarrow R))$

Both  $P \rightarrow Q$  and  $P \rightarrow R$  are false on this row.

$P$  is true on this row while both  $Q$  and  $R$  are false.

But then  $P \rightarrow (Q \vee R)$  is false on this row.

Therefore, in every row where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false,  $P \rightarrow (Q \vee R)$  is also false.

# From informal to formal

We show that  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$ .

Assume  $\neg((P \rightarrow Q) \vee (P \rightarrow R))$

Then we have  $\neg(P \rightarrow Q)$  and  $\neg(P \rightarrow R)$

$P$  is true on this row while both  $Q$  and  $R$  are false.

But then  $P \rightarrow (Q \vee R)$  is false on this row.

Therefore, in every row where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false,  $P \rightarrow (Q \vee R)$  is also false.

# From informal to formal

We show that  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$ .

Assume  $\neg((P \rightarrow Q) \vee (P \rightarrow R))$

Then we have  $\neg(P \rightarrow Q)$  and  $\neg(P \rightarrow R)$

Therefore  $P$ ,  $\neg Q$ , and  $\neg R$

But then  $P \rightarrow (Q \vee R)$  is false on this row.

Therefore, in every row where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false,  $P \rightarrow (Q \vee R)$  is also false.



# From informal to formal

We show that  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$ .

Assume  $\neg((P \rightarrow Q) \vee (P \rightarrow R))$

Then we have  $\neg(P \rightarrow Q)$  and  $\neg(P \rightarrow R)$

Therefore  $P$ ,  $\neg Q$ , and  $\neg R$

So  $\neg(P \rightarrow (Q \vee R))$

Therefore, in every row where  $(P \rightarrow Q) \vee (P \rightarrow R)$  is false,  $P \rightarrow (Q \vee R)$  is also false.

# From informal to formal

We show that  $P \rightarrow (Q \vee R) \vdash (P \rightarrow Q) \vee (P \rightarrow R)$ .

Assume  $\neg((P \rightarrow Q) \vee (P \rightarrow R))$

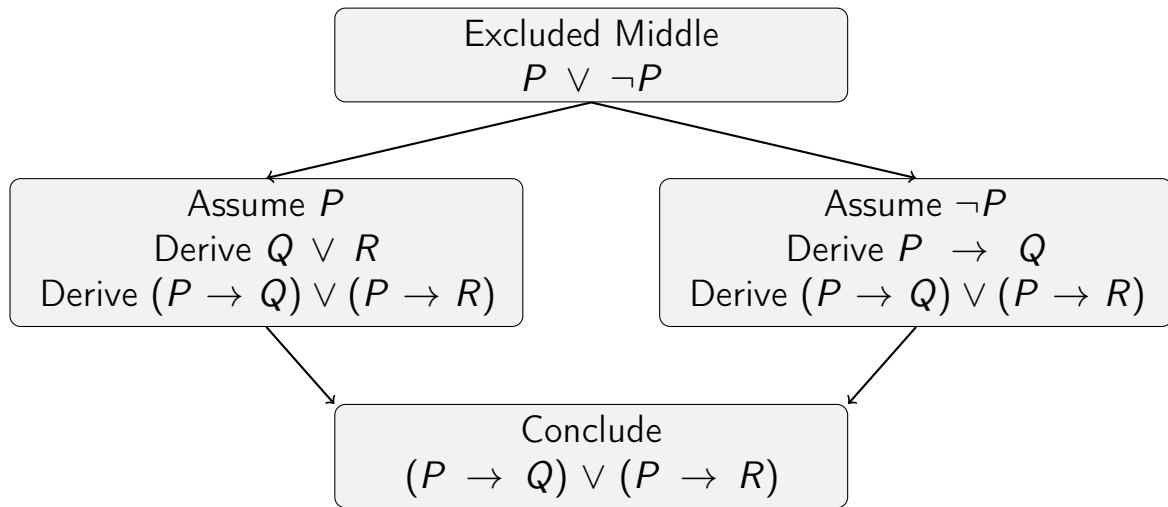
Then we have  $\neg(P \rightarrow Q)$  and  $\neg(P \rightarrow R)$

Therefore  $P$ ,  $\neg Q$ , and  $\neg R$

So  $\neg(P \rightarrow (Q \vee R))$

Hence  $\neg((P \rightarrow Q) \vee (P \rightarrow R)) \vdash \neg(P \rightarrow (Q \vee R))$

# Alternate proof strategy



1	(1)	$(P \rightarrow Q) \rightarrow P$	A
2	(2)	$\neg P$	A
3	(3)	$P$	A
2,3	(4)	$P \wedge \neg P$	2,3 $\wedge I$
5	(5)	$\neg Q$	A
2,3	(6)	$\neg\neg Q$	5,4 RA
2,3	(7)	$Q$	6 DN
2	(8)	$P \rightarrow Q$	3,7 CP
1,2	(9)	$P$	1,8 MP
1,2	(10)	$P \wedge \neg P$	9,2 $\wedge I$
1	(11)	$\neg\neg P$	2,10 RA
1	(12)	$P$	11 DN
$\emptyset$	(13)	$((P \rightarrow Q) \rightarrow P) \rightarrow P$	1,12 CP

# Summary

- With RA, we have completed the set of inference rules for propositional logic.
- These rules are provably **sound**: they do not permit a proof of something that has a truth-table counterexample.
- These rules are provably **complete**: anything semantically valid can be proven.

# Supplemental material

# Redundancies in Our System

- With RA, Modus Tollens (MT) and DN-Intro can be eliminated.
- Example: simulate MT using RA.

1	(1)	$P \rightarrow Q$	A
2	(2)	$\neg Q$	A
3	(3)	$P$	A
1,3	(4)	$Q$	1,3 MP
1,2,3	(5)	$Q \wedge \neg Q$	4,2 $\wedge$ I
1,2	(6)	$\neg P$	3,5 RA

# Simulating DN-Intro

1	(1)	$P$	A
2	(2)	$\neg P$	A
1,2	(3)	$P \wedge \neg P$	1,2 $\wedge I$
1	(4)	$\neg\neg P$	2,3 RA



# Without RA

RA itself can be simulated with other rules.

Suppose  $\Gamma, P \vdash Q \wedge \neg Q$ . Then:

- $\Gamma \vdash P \rightarrow Q$  and  $\Gamma \vdash P \rightarrow \neg Q$ .
- By contraposition:  $\Gamma \vdash \neg Q \rightarrow \neg P$ .
- Hence  $\Gamma \vdash P \rightarrow \neg P$ .
- But  $P \rightarrow \neg P \vdash \neg P$ .

So  $\Gamma \vdash \neg P$ . Still, RA feels more natural and symmetric.

# More difficult proofs

To show:  $\vdash (P \rightarrow Q) \vee (Q \rightarrow P)$

- Strategy 1: Assume  $\neg((P \rightarrow Q) \vee (Q \rightarrow P))$ . Use DM to get  $\neg(P \rightarrow Q)$  and  $\neg(Q \rightarrow P)$ . The former entails  $P$  while the latter entails  $\neg P$ .
- Strategy 2: Derive  $Q \vee \neg Q$ , then argue by cases using positive paradox and negative paradox in turn.