Logic final exam (practice problems)

1. Translate the following sentences into predicate logic. Use a relation symbol Lxy for "x loves y," and use the equality symbol $=$ when appropriate. (You don't need a separate predicate symbol for "x is a person.")

   (a) Everyone loves at least two other people.
   $\forall x \exists y \exists z (Lxy \wedge Lxz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$

   (b) Nobody is loved by more than two people.
   $\neg \exists w \exists x \exists y \exists z (Lxw \wedge Lyw \wedge Lzw \wedge x \neq y \wedge x \neq z \wedge y \neq z)$

   (c) Everyone loves all people who love her.
   $\forall x \forall y (Lyx \rightarrow Lxy)$

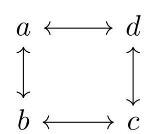
   (d) There is no one who is loved by both Anne and Bente.
   $\neg \exists x (Lax \wedge Lbx)$

   (e) Bente loves nobody but Anne and Cordelia.
   $Lba \wedge Lbc \wedge \forall x (Lbx \rightarrow (x = a \vee x = c))$

   (f) If two people don't love each other, then at least they love all the same people.
   $\forall x \forall y (\neg Lxy \rightarrow \forall z (Lxz \leftrightarrow Lyz))$

2. Provide a model/interpretation to show that these six sentences are consistent. (It would suffice to draw an arrow diagram, or to write a domain set and extensions of the relevant symbols.)

   Consider the following picture:

   where an arrow indicates that the relation Lxy holds between the two individuals. Clearly everyone loves two other people, and nobody is loved by more than two people. Since all arrows are double-headed, it follows that everyone loves all people who love her. Since Anne loves Bente and Dorte, and Bente loves Anne and Cordelia, nobody is loved by both Anne and Bente. Clearly Bente loves nobody but Anne and Cordelia. Finally, if two people don't love each other, then they are diagonal from each other on the square. But that means that both of them love the other two people. Hence if two people don't love each other, then they love all the same people.

3. Show that Anne and Bente are different people.

   Sketch of proof: From (a) we have $\exists x Lax$, hence $\exists x(Lax \wedge Lax)$. If $a = b$, then $\exists x(Lax \wedge Lbx)$, in contradiction with (d). Therefore, $a \neq b$.

4. Show that Anne and Cordelia are different people.

   Sketch of proof: From (e) we have $\forall x(Lbx \leftrightarrow (x = a \vee x = c))$. If $a = c$ then it would follow that $\forall x(Lbx \leftrightarrow x = a)$, but that contradicts (a), which says that $\exists x \exists y(Lbx \wedge Lby \wedge x \neq y)$. Therefore $a \neq c$.

5. Show that Bente and Cordelia are different people.

   Sketch of proof: By (e), Bente loves nobody but Anne and Cordelia. If $b = c$, then Bente only loves one person besides herself, in contradiction with (a). Therefore, $b \neq c$.

6. Show that there is somebody else besides Anne, Bente, and Cordelia.

   Sketch of proof: By (a), Anne loves at least two other people. Since Bente loves Anne, (c) implies that Anne loves Bente. By (e), Bente loves Cordelia; hence by (d), Anne does not love Cordelia. Therefore, there must be another person besides Anne, Bente, and Cordelia.

7. Show that if Anne loves a person, then that person loves Cordelia.

   Sketch of proof: In the previous argument, we showed that Anne doesn't love Cordelia. By (f), Anne and Cordelia love all the same people. Hence, if Anne loves a person, then Cordelia loves that same person, and by (c), that person loves Cordelia.