Practice Final Exam

Short answer

1. State the Existential Elimination (EE) rule, along with each of its restrictions.

   EE allows us to deduce $\Gamma, \Delta \vdash C$ from $\Gamma \vdash \exists x \phi(x)$ and $\Delta, \phi(c) \vdash C$, provided that c does *not* occur in:

   > **1.** the existential claim $\exists x \phi(x)$ which we seek to 'eliminate' using EE;
   > **2.** the premises Δ on which the proof of the conclusion is based; and
   > **3.** the conclusion C.

2. Complete the following sentence: propositional logic sentences ϕ and ψ are mutually consistent just in case ...

   ...there is a valuation v such that $v(\phi) = 1$ and $v(\psi) = 1$.

3. True or False (explain your answer): Suppose that ϕ and ψ are mutually consistent propositional logic sentences. Could there be a correctly written proof that begins with $\phi \wedge \psi$ and that ends with $\bot$?

   No. The Soundness Theorem shows that if $\phi \wedge \psi \vdash \bot$, then for any valuation v such that $v(\phi \wedge \psi) = 1$ we would also have $v(\bot) = 1$. Since ϕ and ψ are mutually consistent, there is a valuation v such that $v(\phi \wedge \psi) = 1$. But $v(\bot) = 0$ for any valuation. Therefore, there is no proof of $\bot$ from $\phi \wedge \psi$.

4. Grade the following proof.

1	(1)	$p \vee q$	A
2	(2)	p	A
3	(3)	q	A
2,3	(4)	$p \wedge q$	2,3 $\wedge$I
2,3	(5)	p	4 $\wedge$E
1	(6)	p	1,2,2,3,5 $\vee$E

   Lines 1 through 5 of this proof are perfectly correct. But we know that $p \vee q$ does not imply p, so there must be something wrong with line

6. Indeed, the dependency tabulation on line 6 is faulty. To compute the dependencies for $\vee$E, we need to combine the following three sets of dependencies:

1. The dependencies of the disjunction (i.e. of line 1).
2. The dependencies of the first derivation (i.e. of line 2), except for the first assumption (i.e. 2).
3. The dependencies of the second derivation (i.e. of line 5), except for the second assumption (i.e. 3).

But the third set of dependencies includes 2. Thus, line 6 should include dependency on 2.

5. Grade the following proof.

1	(1) $\neg p$	A
2	(2) $\exists x(Fx \wedge \neg Fx)$	A
2	(3) $\neg\neg p$	1,2 RAA
2	(4) p	3 DN
	(5) $\exists x(Fx \wedge \neg Fx) \to p$	2,4 CP

Line 3 is incorrect. The rule RAA requires a contradiction, i.e. a sentence of the form $\phi \wedge \neg\phi$. But line 2 is not a contradiction, it is an existential sentence.

6. A "bad line" in a proof is a line where the sentence on the right is not a logical consequence of its dependencies. Identify all the bad lines in the previous two proofs.

In the first proof, line 6 is bad. In the second proof, none of the lines is "bad" in this sense. For although the rule RAA doesn't allow line 3 as it's written, there is nonetheless a proof of $\neg\neg p$ from $\exists x(Fx \wedge \neg Fx)$. In fact, the proof might run as follows: given $\exists x(Fx \wedge \neg Fx)$, assume $Fa \wedge \neg Fa$. Use RAA to derive $\neg\neg p$, and then use EE to show that $\neg\neg p$ follows from $\exists x(Fx \wedge \neg Fx)$.

Translation

Translate the following sentences into predicate logic notation. You may use the equals sign = as well as the following relation symbols:

$Mx \equiv x$ is male $\quad Pxy \equiv x$ is a parent of $y \quad Axy \equiv x$ adores y

(The domain of quantification is persons — you do not need a predicate symbol for "is a person." For the purposes of this problem, a "child" is anyone who has a parent.)

1. Every man who has a son adores him.

   $\forall x(Mx \to \forall y((My \land Pxy) \to Axy))$

2. Every man who has a daughter adores his daughter's mother.

   $\forall x(Mx \to \forall y((\neg My \land Pxy) \to \exists z(\neg Mz \land Pzy \land Axz)))$

   Some people might think that the phrase, "his daughter's mother" requires a statement not only of existence, but also of uniqueness. Of course, that could be captured by adding a clause to the effect that for any other z', if z' is a mother of y then $z' = z$.

   Some people might also think that this sentence doesn't really imply that his daughter has a mother — but only that *if* she has a mother, then he adores her. To capture that sense, we would use

   $\forall z((\neg Mz \land Pzy) \to Axz)$

3. Everybody adores their own grandchildren.

   $\forall x(\forall y(\exists z(Pxz \land Pzy)) \to Axy)$

4. Every woman adores her brothers' children.

   $\forall x(\neg Mx \to \forall y \forall z(((My \land \exists w(Pwx \land Pwy)) \land Pyz) \to Axz))$

   Here we symbolize "y is a brother of x," by

   $My \land \exists w(Pwx \land Pwy)$

   which some people would say only means that y is a half-brother of x.

5. No man adores children unless he has his own.

   $\forall x(Mx \to (\exists y \exists z(Pzy \land Axy) \to \exists w Pxw))$

6. Someone has no more than two children.

   $\exists w \forall x \forall y \forall z((Pwx \land Pwy \land Pwz) \to (x = y \lor x = z \lor y = z))$

Proofs

Prove the following sequents.

1. $\neg(p \to q) \ \vdash \ (p \wedge \neg q)$

   We won't write out the full proof here, because this is an easy one. The idea of the proof is as follows: from negative paradox, we have $\neg p \vdash p \to q$. Hence $\neg(p \to q) \vdash p$. From positive paradox, we have $q \vdash p \to q$. Hence $\neg(p \to q) \vdash \neg q$.

2. $\exists x(Fx \wedge \forall y(Gy \to Rxy)), \forall x(Fx \to \forall y(Hy \to \neg Rxy)) \ \vdash \ \forall x(Gx \to \neg Hx)$

   Again, we will sketch the idea of the proof. Since the conclusion is a universally quantifier conditional, we assume Ga with the goal of proving $\neg Ha$. Now take a typical disjunct of the first existential sentence, e.g.

   $$Fb \wedge \forall y(Gy \to Rby),$$

   which yields Fb and $\forall y(Gy \to Rby)$. Then use UI on the universal statement to get

   $$Fb \to \forall y(Hy \to \neg Rby).$$

   My MPP we have $\forall y(Hy \to \neg Rby)$. Now instantiate the two universal sentences to get $Ga \to Rba$ and $Ha \to \neg Rba$. Using Ga, which we assumed above, we get Rba, hence $\neg\neg Rba$, hence $\neg Ha$. To complete the proof, we then use CP to get $Ga \to \neg Ha$, and UI to get $\forall x(Gx \to \neg Hx)$.

Metatheory

1. Use proof by induction to show that the connective $\vee$ is not by itself truth-functionally complete (i.e. there is a truth-function that cannot be expressed using only $\vee$).

   We will show that for any formula ϕ built out of a single atomic sentence p and $\vee$, and for any valuation v, if $v(p) = 1$ then $v(\phi) = 1$. We prove it by induction. Base case: if $\phi = p$, then the result is obviously true. Now suppose that the result is true for ϕ and ψ. That is, for any valuation v, if $v(p) = 1$ then $v(\phi) = 1$ and $v(\psi) = 1$. But then for any valuation v, if $v(p) = 1$ then $v(\phi \vee \psi) = \max\{v(\phi), v(\psi)\} = 1$.

2. State precisely what it means to say that the propositional logic inference rules are *sound*. i.e. state the soundness theorem for the propositional calculus. Prove the soundness of Reductio ad Absurdum (RAA).

To say that the propositional logic rules are sound means that if there is a derivation of ψ from $\phi_1, \ldots, \phi_n$, then for any valuation v, if $v(\phi_i) = 1$ for $i = 1, \ldots, n$, then $v(\psi) = 1$. Or in more pedestrian terms, on any row of the truth table for $\phi_1, \ldots, \phi_n$ and ψ, if $\phi_1, \ldots, \phi_n$ are all true, then ψ is also true.

3. True or False (explain your answer): if ϕ is a propositional logic tautology, and ϕ' is a substitution instance of ϕ, then ϕ' is a tautology.

   True. By saying that ϕ' is a substitution instance of ϕ, we mean that they are sentences in the same language Σ, and that ϕ' results by replacing one or more atomic sentences in ϕ with sentences of Σ. Now let v be an arbitrary valuation of Σ, and consider the new valuation v' given by

   $$v'(p) \;=\; v(p'),$$

   where p' is whatever sentence replaces p in ϕ'. Then it's straightforward to verify that $v'(\phi) = v(\phi')$, and since ϕ is a tautology, it follows that $v(\phi') = 1$. Since v is an arbitrary valuation, it follows that ϕ' is a tautology.

4. Give a substitution instance of the following sentence that is a tautology:

   $$(p \wedge q) \vee (\neg p \wedge \neg q)$$

   Explain precisely which substitutions you have performed.

   Let $F(p) = \top$ and $F(q) = \top$. Then $F(p \wedge q) = \top \wedge \top$, and F applied to the original sentence is a tautology.